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Convergence of inexact Newton-like iterations in incremental finite element analysis of elasto-plastic problems

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Abstract

Convergence of two inexact Newton-like methods suitable for application in incremental finite element analysis of problems of elasto-plasticity is investigated by a new technique based on a certain approximation condition. It is shown that the convergence can be controlled by the size of load increments. A numerical example illustrates the developed theoretical results.

1. Introduction

The use of incremental finite element analysis for the numerical solution of problems of solid mechanics with elasto-plastic material behaviour leads to the necessity of solving large nonlinear systems of equations. This is usually done by some Newton-like iterative method, such as the initial or tangent stiffness methods described later in Sections 6 and 7. One of our goals is to clear up the conditions for the convergence of these methods.

The numerical implementation of the Newton-like methods requires the solution of large sparse linear systems. This task can be performed by direct or iterative solvers. In many cases, e.g. typically when solving 3D problems, the use of iterative solvers is more efficient (see e.g. [1,2,8]). But when we use an iterative solver then it is very natural to use inexact versions of the Newton-like algorithms in order to not to waste our effort by solving the linear systems with too high an accuracy. The second goal in this paper is the proof of convergence of the inexact versions of the initial and tangent stiffness iterative methods.

The paper begins with a short description of elasto-plasticity problems. The numerical technique considered here for the solution of these problems, is the incremental finite element method which is described in Section 3. The stress computation within the load steps is assumed to be performed with the aid of the elasto-plastic tangential operator. A regularization of the stress computation procedure is described in Section 4. In this introductory part of our paper, we closely follow the inspiring book [9], but many details can be found also in other publications such as [12,11]. The use of regularization enables us to obtain nonlinear systems with continuous and locally strong monotone operator, see

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Section 5. The results of this section are similar to the results of [9], but we hope that our proofs are found to be somewhat simpler and that we succeeded in removing some drawbacks from [9].

Sections 6 and 7 contain our main convergence results. In the analysis of convergence we exploit a new analysis technique based on certain approximation conditions. This technique, previously used in [4], enables us to prove the convergence under very natural assumptions which concern only the size of the load increments and gives results which are stronger than the results which can be obtained directly from the local monotonicity and Lipschitz continuity.

Finally, the results for standard incremental methods with another type of regularization are discussed and some numerical examples are presented in Section 8.

NOTATION. Throughout the paper, we use the notations

$$\langle u, v \rangle = u^T v, \quad \|u\| = \sqrt{u^T u}$$

for $u \in R^m$, $m \geq 1$. Moreover, a dot denotes a derivative with respect to t , e.g.

$$\dot{u} = \frac{du}{dt}.$$

2. Elasto-plasticity

Let us consider a body $\Omega \subset R^3$ whose internal state after loading is described by the *displacement*

$$u = u(x) = (u_1(x), u_2(x), u_3(x)), \quad x \in \Omega$$

and the vectors

$$e = e(u, x) = (e_{11}, e_{22}, e_{33}, e_{12}, e_{21}, e_{23}, e_{32}, e_{13}, e_{31})^T$$

$$\sigma = \sigma(x) = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{21}, \sigma_{23}, \sigma_{32}, \sigma_{13}, \sigma_{31})^T$$

which contain all components of the *small strain tensor* and the *Cauchy stress tensor*, respectively. Here,

$$e, \sigma \in S = \{v \in R^9 : v_4 = v_5, v_6 = v_7, v_8 = v_9\}, \quad (1)$$

$$e_{ij} = e_{ij}(u, x) = \frac{1}{2} \frac{\partial u_i}{\partial x_j}(x) + \frac{1}{2} \frac{\partial u_j}{\partial x_i}(x). \quad (2)$$

The *standard model of elasto-plasticity* is then defined by the relations

$$e = e^e + e^p, \quad e = e(u),$$

$$\sigma = D e^e,$$

$$P(\sigma, \kappa) \leq 0,$$

$$\Delta e^p = \gamma p, \quad p = \frac{\partial p}{\partial \sigma} \in S,$$

$$\Delta \kappa = \gamma z$$

where

- e^e, e^p are, respectively, elastic and plastic part of the strain vector e , which is supposed to be compatible with some displacement u ,
- D is a 9×9 matrix which defines a linear relation between σ and e^e , D is symmetric and positive definite in S ,
- $P = P(\sigma, \kappa)$ is a function defining the yield surface $P = 0$,
- $\kappa \in Z$ is a scalar or a vector of hardening parameters, $Z = R^k$, $k \geq 1$, is the space of hardening parameters,
- Δe^p is an increment of plastic deformations,

- γ is a plastic multiplier (associated plasticity is assumed),
- $\Delta\kappa$ is an increment of hardening parameters,
- $z \in Z$ is a scalar or a vector depending on the type of the adopted hardening rule.

All the above quantities depend also on $x \in \Omega$ as both the material and the stress–strain state are x dependent. The details can be found e.g. in [11,9].

To simplify the presentation, we assume that P is everywhere differentiable and denote by p, q the gradients of P corresponding to σ and κ ,

$$p = \frac{\partial P}{\partial \sigma} \in S, \quad q = \frac{\partial P}{\partial \kappa} \in Z.$$

We also assume the validity of the inequalities

$$q^T z < 0, \quad 0 < \frac{p^T Dp}{p^T Dp - q^T z} \leq \nu_0 < 1 \quad (3)$$

EXAMPLE: von Mises plasticity with strain hardening.

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{bmatrix},$$

$$D_2 = \text{diag}(2\mu, 2\mu, 2\mu, 2\mu, 2\mu, 2\mu)$$

where λ, μ are the Lamé moduli and

$$P(\sigma, \kappa) = \bar{s} - H(\kappa),$$

$$\bar{s} = \sqrt{\frac{3}{2} s^T s}, \quad \kappa = \sqrt{\frac{2}{3}} \|\dot{e}^p\|,$$

where $s \in S$ is the deviator of the stress σ , $s_{ii} = \sigma_{ii} - \sigma_0/3$, $\sigma_0 = \sigma_{11} + \sigma_{22} + \sigma_{33}$, $s_{ij} = \sigma_{ij}$ for $i \neq j$.

The function H is derived from experimental results, e.g. from uniaxial tests. We shall assume

$$\frac{dH}{d\kappa} = H' \geq c_0 > 0.$$

For the described plasticity model, we further obtain

$$p = \frac{\partial P}{\partial \sigma} = \frac{\partial P}{\partial s} = \frac{3}{2} \frac{s}{\bar{s}},$$

$$\dot{\kappa} = \sqrt{\frac{2}{3}} \|\dot{e}^p\| = \gamma z, \quad z = \sqrt{\frac{2}{3}} \|p\| = 1,$$

$$s^T Ds = 2\mu s^T s + \lambda (s_{11} + s_{22} + s_{33})^2 = 2\mu \frac{2}{3} \bar{s}^2,$$

$$p^T Dp = 3\mu,$$

$$q = \frac{\partial P}{\partial \kappa} = -H', \quad q^T z = qz = -H' \leq -c_0 < 0,$$

$$\frac{p^T Dp}{p^T Dp - q^T z} = \frac{3\mu}{3\mu + H'} \leq \frac{3\mu}{3\mu + c_0} = \nu_0 < 1.$$

3. Incremental finite element analysis

Let us consider the body Ω fixed in the part Γ_0 of its boundary $\partial\Omega$ and loaded by the volume force f_v in Ω and by the surface force f_s on the part Γ_1 of $\partial\Omega$.

To formulate the problem of elasto-plasticity, we can follow the history of loading. For this we introduce a continuation parameter $t \in [0, T]$ and state the problem: find

$$u = u(x, t), \quad \sigma = \sigma(x, t), \quad \kappa = \kappa(x, t)$$

such that

$$\int_{\Omega} \dot{\sigma}^T e(v) \, dx = (\dot{F}, v) \quad \forall v \in V, \quad t \in (0, T]$$

$$\dot{\sigma} = D_{ep}(\sigma, \kappa, \dot{e}) \dot{e}$$

$$\dot{\kappa} = G(\sigma, \kappa, \dot{e}) \dot{e}$$

$$\dot{e} = e(\dot{u})$$

$$\dot{u} \in V$$

with initial conditions

$$u(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad \kappa(x, 0) = 0 \quad \forall x \in \Omega.$$

Here,

$$V = \{v = (v_1, v_2, v_3) : v_i \in H^1(\Omega), v = 0 \text{ on } \Gamma_0\},$$

$$(\dot{F}, v) = \int_{\Omega} \dot{f}_v^T v \, dx + \int_{\Gamma_1} \dot{f}_s^T v \, ds.$$

Moreover, we use the incremental constitutive relations

$$D_{ep}(\sigma, \kappa, \dot{e}) = D - \rho(\sigma, \kappa, \dot{e}) D_p(\sigma, \kappa), \quad (4)$$

$$G(\sigma, \kappa, \dot{e}) = \rho(\sigma, \kappa, \dot{e}) G_p(\sigma, \kappa) \quad (5)$$

with $\rho = \rho(\sigma(x), \kappa(x), \dot{e}(x))$,

$$\rho(\sigma, \kappa, \dot{e}) = \begin{cases} 0 & \text{if } p^T D \dot{e} \leq 0 \\ 1 & \text{if } p^T D \dot{e} > 0 \end{cases}$$

$$D_p = \frac{D p p^T D}{p^T D p - q^T z}, \quad G_p = \frac{z p^T D}{p^T D p - q^T z}. \quad (6)$$

For the numerical solution, we consider a discretization given by using finite increments of load and replacing the space V by some finite element subspace V_h . Thus, we approximate $u(x, t_k)$, $\sigma(x, t_k)$, $\kappa(z, t_k)$, $t_k = k \Delta t$, $k = 0, 1, \dots, T/\Delta t$, by

$$u_h^k \in V_h, \quad \sigma_h^k, \quad \kappa_h^k$$

which can be computed by the *explicit incremental finite element algorithm* described in the following steps.

Initial step. Put $u_h^0 = 0$, $\sigma_h^0 = 0$, $\kappa_h^0 = 0$.

Load steps. ($k = 0, \dots, T/\Delta t - 1$):

Given u_h^k , σ_h^k , κ_h^k , compute Δu_h , $\Delta \sigma_h$, $\Delta \kappa_h$

$$\int_{\Omega} \Delta \sigma_h^T e(v_h) \, dx = (\Delta F^k, v_h) \quad \forall v_h \in V_h, \quad (7)$$

$$\Delta \sigma_h = D_{ep}(\sigma_h^k, \kappa_h^k, \Delta e) \Delta e_h, \quad (8)$$

$$\Delta \kappa_h = G(\sigma_h^k, \kappa_h^k, \Delta e) \Delta e, \quad (9)$$

$$\Delta e_h = e(\Delta u_h), \quad \Delta u_h \in V_h,$$

$$(\Delta F^k, v_h) = \int_{\Omega} (\Delta f_v^k)^T v_h \, dx + \int_{\Gamma_1} (\Delta f_s^k)^T v_h \, ds ,$$

$$\Delta f_i^k(x) = f_i(x, t_{k+1}) - f_i(x, t_k) , \quad i = v, s .$$

Put:

$$u_h^{k+1} = u_h^k + \Delta u_h , \quad \sigma_h^{k+1} = \sigma_h^k + \Delta \sigma_h , \quad \kappa_h^{k+1} = \kappa_h^k + \Delta \kappa_h .$$

End of the load step.

For the stress computation within the incremental algorithm, the state multiplier ρ can be defined by the conditions

$$\rho(\sigma_h^k, \kappa_h^k, \Delta e_h) = \begin{cases} 0 & \text{if } P(\sigma_h^k + D \Delta e_h, \kappa_h^k) \leq 0 \\ 1 & \text{if } P(\sigma_h^k + D \Delta e_h, \kappa_h^k) > 0 \end{cases}$$

In each loading step, we have to solve a nonlinear problem which can be formulated as a system of nonlinear equations. To this end, we shall use an isomorphism

$$\beta : R'' \rightarrow V_h$$

and denote

$$B_u = e(\beta u) \quad \text{for } u \in R'' .$$

DEFINITION. For $\sigma \in S$, $\kappa \in Z$ and $w \in R''$ fixed, we define two mappings $A_{ep}(\sigma, \kappa, w)$ and A_e both from R'' to R'' and the vector $\Delta f^k \in R''$ by the following identities

$$\langle A_{ep}(\sigma, \kappa, w)u, v \rangle = \int_{\Omega} (D_{ep}(\sigma, \kappa, Bw)Bu)^T Bv \, dx , \quad (10)$$

$$\langle A_e u, v \rangle = \int_{\Omega} (DBu)^T Bv \, dx , \quad (11)$$

$$\langle \Delta f^k, v \rangle = (\Delta F^k, \beta v) \quad (12)$$

which should be valid for all $v \in R''$.

DEFINITION. Since A_e is symmetric and positive definite, we can introduce the following two norms in R'' ,

$$\|u\|_E = \sqrt{\langle A_e u, u \rangle} = \|A_e^{1/2} u\| ,$$

$$\|u\|_{-E} = \sqrt{\langle A_e^{-1} u, u \rangle} = \|A_e^{-1/2} u\| .$$

In each step of the explicit incremental finite element algorithm we must solve the nonlinear equation

$$\mathcal{F}(\Delta u) = \Delta f^k \quad (13)$$

with the nonlinear operator

$$\mathcal{F}(\Delta u) = A_{ep}(\sigma^k, \kappa^k, \Delta u) \Delta u . \quad (14)$$

4. Regularized stress computation

The just introduced mapping \mathcal{F} is not continuous. It can be seen if we take

$$u \in R'' , \quad \rho(\sigma, \kappa, u) = 1 ,$$

$$u_k \in R'' , \quad \rho(\sigma, \kappa, u_k) = 0, \quad u_k \rightarrow u ,$$

because then

$$\mathcal{F}(u) - \mathcal{F}(u_k) \rightarrow A_{ep}(\sigma, \kappa, u)u - A_e u \neq 0.$$

This discontinuity is a reason for introducing a *regularized stress computation procedure* with smoothed transition from the elastic to plastic state (see also [9]). The regularized model is more suitable for the numerical solution but we believe that it can also favourably influence the accuracy of the numerical solution.

The regularization can be obtained by replacing the discontinuous multiplier ρ by a continuous $\bar{\rho}$ defined in the following way.

DEFINITION. Let $\delta > 0$ be a constant. Then we define

$$\bar{\rho} = \bar{\rho}_\delta(\sigma(x), \kappa(x), \Delta e(x)),$$

$$\bar{\rho}_\delta(\sigma, \kappa, \Delta e) = \begin{cases} 0 & \text{if } P = P(\sigma + D \Delta e, \kappa) \leq -\delta \\ 1 + P/\delta & \text{if } -\delta < P \leq 0 \\ 1 & \text{if } P > 0 \end{cases}$$

The use of $\bar{\rho}$ leads to the incremental finite element method with the nonlinear systems

$$\bar{\mathcal{F}}(\Delta u) = \Delta f^k, \quad \Delta u, \Delta f^k \in R^n, \quad (15)$$

where for all $v \in R^n$

$$\langle \bar{\mathcal{F}}(\Delta u), v \rangle = \int_{\Omega} [(D - \bar{\rho}(\sigma^k, \kappa^k, B \Delta u) D_p) B \Delta u]^T B v \, dx. \quad (16)$$

LEMMA 1. For $x \in \Omega$ we can define a norm in S by

$$\|s\|_D = \|s\|_{D(x)} = \sqrt{(D(x)s)^T s}.$$

For $\sigma \in S$, $\kappa \in Z$, $x \in \Omega$, $u, v \in R^n$, we shall denote

$$\bar{\rho}(x, u) = \bar{\rho}_\delta(\sigma, \kappa, Bu(x)),$$

$$\bar{\rho}(x, v) = \bar{\rho}_\delta(\sigma, \kappa, Bv(x))$$

and we shall suppose that $p = \partial P / \partial \sigma$ is uniformly bounded with respect to σ, κ , i.e.

$$\|p\|_D \leq C.$$

Then

$$|\bar{\rho}(x, u) - \bar{\rho}(x, v)| \leq \frac{C}{\delta} \|Bu(x) - Bv(x)\|_D. \quad (17)$$

Moreover, there exists a constant K such that

$$\operatorname{esssup}_{x \in \Omega} |\bar{\rho}(x, u) - \bar{\rho}(x, v)| \leq K \|u - v\|_E \quad (18)$$

PROOF.

$$\begin{aligned} |\bar{\rho}(x, u) - \bar{\rho}(x, v)| &\leq \delta^{-1} |P(\sigma + DBu(x), \kappa) - P(\sigma + DBv(x), \kappa)| \\ &= \delta^{-1} \left| \left[\frac{\partial P}{\partial \sigma}(\sigma^*, \kappa) \right]^T D[Bu(x) - Bv(x)] \right| \\ &\leq \delta^{-1} C \|Bu(x) - Bv(x)\|_D. \end{aligned}$$

The estimate (18) now follows from the fact that both

$$\operatorname{esssup}_{x \in \Omega} \|w(x)\|_{D(x)}$$

and

$$\|w\|_E = \left\{ \int_{\Omega} \|w(x)\|_{D(x)}^2 dx \right\}^{1/2}$$

are equivalent norms on the finite dimensional space of functions

$$w: \Omega \rightarrow S, \quad w = Bv, \quad v \in R^n. \quad \square$$

NOTE 1. For the von Mises plasticity (see Example in Section 2) we have $\|p\|_D = \sqrt{3\mu}$ where μ is the Lamé modulus (shear modulus).

NOTE 2. In the case of linear finite elements, the function $w = Bv$, $v \in R^n$ is constant on the individual elements. Then we have

$$\operatorname{esssup}_{x \in \Omega} \|w(x)\|_D \leq \gamma^{-1/2} \|w\|_E$$

where γ is minimal element volume. The constant K then increases with the grid refinement, $K = O(h^{-3/2})$.

5. Properties of \mathcal{F}

In this Section, we shall prove existence and uniqueness of the solution of the nonlinear system (15) as a consequence of local monotonicity and Lipschitz continuity of \mathcal{F} .

LEMMA 2. Let ν_0 be the constant from (3), $x \in \Omega$ and let

$$|s|_{D_p} = |s|_{D_p(x)} = \sqrt{(D_p(x)s)^T s} \quad (19)$$

be a seminorm in S . Then

$$|s|_{D_p(x)} \leq \sqrt{\nu_0} \|s\|_{D(x)} \quad \text{for all } s \in S. \quad (20)$$

PROOF. The definition of D_p yields

$$(D_p s)^T s = \frac{(p^T Ds)^2}{p^T Dp - q^T z} \leq \frac{(p^T Dp) \cdot (s^T Ds)}{p^T Dp - q^T z} \leq \nu_0 (Ds)^T s$$

for all $s \in S$. \square

LEMMA 3. For all $v, w \in R^n$, we have

$$\int_{\Omega} |(D_p Bv)^T Bw| dx \leq \nu_0 \|v\|_E \cdot \|w\|_E. \quad (21)$$

PROOF.

$$\begin{aligned} \int_{\Omega} |(D_p Bv)^T Bw| dx &\leq \int_{\Omega} |Bv|_{D_p} |Bw|_{D_p} dx \\ &\leq \nu_0 \int_{\Omega} \|Bv\|_D \|Bw\|_D dx \\ &\leq \nu_0 \left\{ \int_{\Omega} \|Bv\|_D^2 dx \right\}^{1/2} \left\{ \int_{\Omega} \|Bw\|_D^2 dx \right\}^{1/2} \\ &= \nu_0 \|v\|_E \|w\|_E \quad \square \end{aligned}$$

Now, let us consider the nonlinear mapping \mathcal{F} . For $u, v, w \in R^n$, we obtain

$$\begin{aligned}
\langle \bar{\mathcal{F}}(u) - \bar{\mathcal{F}}(v), w \rangle &= \int_{\Omega} [DB(u - v)]^T Bw \, dx \\
&\quad - \int_{\Omega} [\bar{\rho}(x, u)D_p Bu - \bar{\rho}(x, v)D_p Bv]^T Bw \, dx \\
&= \langle A_e(u - v), w \rangle - I_1 - I_2,
\end{aligned} \tag{22}$$

where

$$I_1 = \int_{\Omega} [\bar{\rho}(x, u)D_p B(u - v)]^T Bw \, dx \tag{23}$$

$$I_2 = \int_{\Omega} [\bar{\rho}(x, u) - \bar{\rho}(x, v)](D_p Bv)^T Bw \, dx. \tag{24}$$

According to Lemmas 3 and 1, we obtain

$$|I_1| \leq \nu_0 \|u - v\|_E \|w\|_E \tag{25}$$

$$|I_2| \leq \nu_0 K \|v\|_E \|u - v\|_E \|w\|_E. \tag{26}$$

The above estimates imply that $\bar{\mathcal{F}}$ is Lipschitz continuous and strongly monotone in any ball $B_\alpha = \{v \in R^n, \|v\|_E \leq \alpha\}$ if $\zeta = \nu_0 + \nu_0 K \alpha < 1$.

As an consequence, we can prove existence and uniqueness of the solution of the system (15) under assumption that the load increment Δf^k is sufficiently small.

THEOREM 1. *Let α be such that $\zeta = \nu_0 + K\nu_0\alpha < 1$ for ν_0, K defined in (3) and (18), respectively. Denote $m = 1 - \zeta$, $M = 1 + \zeta$. Moreover, let $\omega \in (0, 2m/M^2)$ which gives that*

$$c^2 = 1 - 2m\omega + M^2\omega^2 < 1.$$

Further, let the load increment Δf^k be sufficiently small, e.g.

$$\|\Delta f^k\|_{-E} \leq \frac{1 - c}{\omega} \alpha.$$

Then the iterations

$$\Delta u^{i+1} = \Delta u^i + \omega A_e^{-1}(\Delta f^k - \bar{\mathcal{F}}(\Delta u^i)), \quad \Delta u^0 = 0 \tag{27}$$

converge to the unique solution of the equation $\bar{\mathcal{F}}(\Delta u) = \Delta f^k$ in the ball $B_\alpha = \{v \in R^n, \|v\|_E \leq \alpha\}$.

PROOF. Let $\Delta u^{i-1} \in B_\alpha$, then

$$\begin{aligned}
\|\Delta u^{i+1} - \Delta u^i\|_E^2 &= \|\Delta u^i - \Delta u^{i-1}\|_E^2 - 2\omega \langle \Delta u^i - \Delta u^{i-1}, \bar{\mathcal{F}}(\Delta u^i) - \bar{\mathcal{F}}(\Delta u^{i-1}) \rangle \\
&\quad + \omega^2 \|A_e^{-1}[\bar{\mathcal{F}}(\Delta u^i) - \bar{\mathcal{F}}(\Delta u^{i-1})]\|_E^2 \\
&\leq c^2 \|\Delta u^i - \Delta u^{i-1}\|_E^2
\end{aligned} \tag{28}$$

because of the local monotonicity and Lipschitz continuity of $\bar{\mathcal{F}}$ which can be obtained from (22)–(26), i.e.

$$\langle \bar{\mathcal{F}}(\Delta u^i) - \bar{\mathcal{F}}(\Delta u^{i-1}), \Delta u^i - \Delta u^{i-1} \rangle \geq m \|\Delta u^i - \Delta u^{i-1}\|_E^2, \tag{29}$$

$$\begin{aligned}
\|A_e^{-1}[\bar{\mathcal{F}}(\Delta u^i) - \bar{\mathcal{F}}(\Delta u^{i-1})]\|_E &\leq \|A_e^{-1/2}[\bar{\mathcal{F}}(\Delta u^i) - \bar{\mathcal{F}}(\Delta u^{i-1})]\| \\
&= \sup_{w \neq 0} \frac{\langle \bar{\mathcal{F}}(\Delta u^i) - \bar{\mathcal{F}}(\Delta u^{i-1}), A_e^{-1/2} w \rangle}{\|w\|} \\
&\leq \sup_{w \neq 0} \frac{\langle \bar{\mathcal{F}}(\Delta u^i) - \bar{\mathcal{F}}(\Delta u^{i-1}), w \rangle}{\|w\|_E} \\
&= M \|\Delta u^i - \Delta u^{i-1}\|_E.
\end{aligned} \tag{30}$$

Further, we can prove that the iterations Δu^i belong to B_α for all $i \geq 0$. Clearly,

$$\Delta u^0 = 0, \quad \text{i.e. } \Delta u^0 \in B_\alpha,$$

$$\|\Delta u^1\|_E = \|\omega A_e^{-1} \Delta f^k\|_E = \omega \|\Delta f^k\|_{-E} \leq \alpha, \quad \text{i.e. } \Delta u^1 \in B_\alpha.$$

Finally, for $i > 1$, we have

$$\begin{aligned}
\|\Delta u^i\|_E &\leq \|\Delta u^i - \Delta u^{i-1}\|_E + \dots + \|\Delta u^1 - \Delta u^0\|_E \\
&\leq (c^{i-1} + \dots + 1) \|\Delta u^1 - \Delta u^0\|_E \leq \frac{1}{1-c} \omega \|\Delta f^k\|_{-E} \leq \alpha.
\end{aligned}$$

The rest of the proof is standard. It is possible to show that $\{\Delta u^i\}$ fulfils the Cauchy condition and therefore that it has a limit Δu in B_α . This limit is a solution of Eq. (15). According to the monotonicity of $\bar{\mathcal{F}}$ in B_α , the solution in B_α is unique. \square

6. Inexact initial stiffness (IIS) algorithm

For the solution of the nonlinear system (15), we can use a Newton-like iterative algorithm whose iteration $\Delta u^i \rightarrow \Delta u^{i+1}$ is given by the following relations:

$$\Delta u^{i+1} = \Delta u^i + \omega d^i, \tag{31}$$

$$A_e d^i = \Delta f^k - \bar{\mathcal{F}}(\Delta u^i) \equiv -R(\Delta u^i) \tag{32}$$

where $\omega > 0$ is a parameter.

This algorithm, often called the *initial stiffness algorithm*, is equal to the iterative method (27) used in Theorem 1. In this section, we shall consider the inexact variant of this algorithm and we shall prove in Theorem 2 that the algorithm converges even without damping, i.e. with $\omega = 1$. The convergence will be guaranteed only by assumption that the load step is sufficiently small.

In the *inexact variant* of this algorithm, it suffices to take d^i which satisfy Eq. (32) only approximately, e.g. for which we have

$$\|A_e d^i + R(\Delta u^i)\|_{-E} \leq \eta \|R(\Delta u^i)\|_{-E} \tag{33}$$

for some $\eta < 1$.

Note that $-E$ norm of the residual is in fact the energy E norm of the error. Thus, the condition (33) will be fulfilled if, e.g. d^i is a sufficiently accurate approximate solution of (32) obtained by the conjugate gradient method.

THEOREM 2. Let $\alpha > 0$ be such that $\zeta = \nu_0 + \nu_0 K \alpha < 1$ for ν_0, K defined in (3) and (18), respectively. Moreover, let η from (33) and $\omega > 0$ be such that

$$c = |1 - \omega| + \omega(\zeta + \eta + \zeta\eta) < 1. \tag{34}$$

Further, let $\Delta f^k, \Delta u^0 \in R''$ be such that

$$\|\Delta u^0\|_E + \frac{\omega(1+\eta)}{1-c} \|\Delta f^k - \bar{\mathcal{F}}(\Delta u^0)\|_{-E} \leq \alpha, \tag{35}$$

i.e. for example, $\Delta u^0 = 0$, $\|\Delta f^k\|_{-E} \leq (1-c)\omega^{-1}(1+\eta)^{-1}\alpha$.

Then the inexact initial stiffness algorithm (31), (32), (33) converges to the unique solution Δu of the system (15) in the ball $B_\alpha = \{v \in R^n, \|v\|_E \leq \alpha\}$.

NOTE 3. The condition (34) can be satisfied if $\zeta + \eta + \zeta\eta < 1$. Then the optimal values are $\omega = 1$, $c = \zeta + \eta + \zeta\eta < 1$.

The condition (35) will be satisfied if the load increments are sufficiently small.

Note also that $(1-c)\omega^{-1} = 1 - \zeta - \eta - \zeta\eta$ for $0 < \omega \leq 1$ and therefore taking small ω does not decrease the restriction on the size of the load increments.

PROOF. Let $u, v \in R^n$, then

$$\begin{aligned} |\langle \bar{\mathcal{F}}(u) - \bar{\mathcal{F}}(v) - A_e(u-v), w \rangle| &= \left| \int_{\Omega} [\bar{\rho}(x, u) D_p B u - \bar{\rho}(x, v) D_p B v]^T B w \, dx \right| \\ &\leq \zeta_v \|u - v\|_E \|w\|_E, \quad \zeta_v = \nu_0 + \nu_0 K \|v\|_E. \end{aligned} \quad (36)$$

This estimate gives the following *approximation condition*

$$\begin{aligned} \|\bar{\mathcal{F}}(u) - \bar{\mathcal{F}}(v) - A_e(u-v)\|_{-E} &= \sup_{w \neq 0} |\langle A_e^{-1/2} [\bar{\mathcal{F}}(u) - \bar{\mathcal{F}}(v) - A_e(u-v)], w \rangle| \|w\|^{-1} \\ &= \sup_{w \neq 0} |\langle \bar{\mathcal{F}}(u) - \bar{\mathcal{F}}(v) - A_e(u-v), A_e^{-1/2} w \rangle| \|w\|^{-1} \\ &\leq \zeta_v \sup_{w \neq 0} \|u - v\|_E \|A_e^{-1/2} w\|_E \|w\|^{-1} \\ &= \zeta_v \sup_{w \neq 0} \|A_e(u-v)\|_{-E} \|w\| \|w\|^{-1} = \zeta_v \|A_e(u-v)\|_{-E} \end{aligned} \quad (37)$$

As a consequence of this approximation condition, we obtain the estimate

$$\begin{aligned} \|R(\Delta u^{i+1})\|_{-E} &= \|R(\Delta u^i + \omega d^i)\|_{-E} \\ &\leq \|(1-\omega)R(\Delta u^i)\|_{-E} + \|\omega[R(\Delta u^i) + A_e d^i]\|_{-E} \\ &\quad + \|R(\Delta u^i + \omega d^i) - R(\Delta u^i) - \omega A_e d^i\|_{-E} \\ &\leq (|1-\omega| + \omega\eta) \|R(\Delta u^i)\|_{-E} + \omega \zeta_{\Delta u^i} \|A_e d^i\|_{-E} \\ &\leq (|1-\omega| + \omega\eta + \omega \zeta_{\Delta u^i} (1+\eta)) \|R(\Delta u^i)\|_{-E} \end{aligned} \quad (38)$$

By induction, we can now prove that under the assumption (35) the inexact initial stiffness algorithm procedures the sequence $\{\Delta u^i\}$ for which

$$\|\Delta u^i\|_E \leq \alpha, \quad (39)$$

$$\|R(\Delta u^{i+1})\|_{-E} \leq c \|R(\Delta u^i)\|_{-E}. \quad (40)$$

First, from (35), it follows that

$$\|\Delta u^0\|_E \leq \alpha$$

thus, $\zeta_{\Delta u^0} \leq \zeta$ and from (38) we obtain (40) for $i = 0$. Moreover, (35) gives

$$\begin{aligned} \|\Delta u^1\|_E &\leq \|\Delta u^0\|_E + \omega \|d^0\|_E \leq \|\Delta u^0\|_E + \omega \|A_e d^0\|_{-E} \\ &\leq \|\Delta u^0\|_E + (1+\eta)\omega \|R(\Delta u^0)\|_{-E} \\ &\leq \|\Delta u^0\|_E + (1+\eta)\omega \|R(\Delta u^0)\|_{-E} \leq \alpha. \end{aligned}$$

Second, let (39), (40) be valid for all $i \leq j$. Then

$$\begin{aligned}
\|\Delta u^{j+1}\|_E &\leq \|\Delta u^j\|_E + (1+\eta)\omega \|R(\Delta u^j)\|_{-E} \\
&\leq \|\Delta u^0\|_E + (1+\eta)\omega(1+c+\dots+c^j)\|R(\Delta u^0)\|_{-E} \\
&\leq \|\Delta u^0\|_E + \frac{(1+\eta)\omega}{1-c} \|R(\Delta u^0)\|_{-E} \leq \alpha.
\end{aligned}$$

Thus, according to (38), the conditions (39), (40) hold also for all $i \leq j+1$.

Finally, from (39), (40), it follows that

- there is a convergent subsequence $\Delta u^{i_k} \rightarrow \Delta u \in B_\alpha$,
- by continuity of $\tilde{\mathcal{F}}$ and R , we obtain $R(\Delta u) = 0$.

From the monotonicity (29) of $\tilde{\mathcal{F}}$ in B_α , we obtain

$$\begin{aligned}
(1-\zeta)\|\Delta u^i - \Delta u\|_E^2 &\leq \langle \Delta u^i - \Delta u, \tilde{\mathcal{F}}(\Delta u^i) - \tilde{\mathcal{F}}(\Delta u) \rangle \\
&= \langle A_e^{1/2}(\Delta u^i - \Delta u), A_e^{-1/2}[R(\Delta u^i) - R(\Delta u)] \rangle \\
&\leq \|\Delta u^i - \Delta u\|_E \|R(\Delta u^i)\|_{-E}.
\end{aligned}$$

Thus, the monotonicity of $\tilde{\mathcal{F}}$ in B_α gives the convergence of Δu^i to Δu and the uniqueness of the solution of the system (15) in B_α . \square

NOTE 4. The estimates developed above can be somewhat pessimistic for problems for which the plastic behaviour is restricted to a small part of the domain Ω .

7. Inexact tangential stiffness (ITS) algorithm

Another Newton-like algorithm for the iterative solution of the nonlinear system (15) arises by replacing A_e in the correction equation (32) by

$$A_{ep}(\Delta u^i) = A_{ep}(\sigma^k, \kappa^k, \Delta u^i).$$

This leads to the *tangential stiffness algorithm* whose iteration $\Delta u^i \rightarrow \Delta u^{i+1}$ is given by the relations

$$\Delta u^{i+1} = \Delta u^i + \omega d^i, \quad (41)$$

$$A_{ep}(\Delta u^i)d^i = \Delta f^k - \tilde{\mathcal{F}}(\Delta u^i) \equiv -R(\Delta u^i) \quad (42)$$

where $\omega > 0$ is a parameter.

The inexact version of this algorithm will be obtained again by permitting that (42) is satisfied only approximately, e.g. that

$$\|A_{ep}(\Delta u^i)d^i + R(\Delta u^i)\|_{-E} \leq \eta \|R(\Delta u^i)\|_{-E} \quad (43)$$

for some $\eta < 1$.

THEOREM 3. Let $\alpha > 0$ be such that $\bar{\zeta} = \nu_0(1-\nu_0)^{-1}K\alpha < 1$ for ν_0 , K defined in (3) and (18), respectively. Moreover, let η from (43) and $\omega > 0$ be such that

$$c = |1-\omega| + \omega(\bar{\zeta} + \eta + \bar{\zeta}\eta) < 1. \quad (44)$$

Further, let $\Delta f^k, \Delta u^0 \in R^n$ fulfil a similar condition as in Theorem 2, i.e.

$$\|\Delta u^0\|_E + \frac{\omega(1+\eta)}{(1-\nu_0)(1-c)} \|\Delta f^k - \tilde{\mathcal{F}}(\Delta u^0)\|_{-E} \leq \alpha. \quad (45)$$

Then the inexact tangential stiffness algorithm (41), (42), (43) converges to the unique solution of the system (15) in the ball $B_\alpha = \{v \in R^n, \|v\|_E \leq \alpha\}$.

PROOF. We shall proceed similarly as in the proof of Theorem 2. First, we derive the approximation condition.

$$|\langle \bar{\mathcal{F}}(\Delta u^i + \omega d^i) - \bar{\mathcal{F}}(\Delta u^i) - A_{ep}(\Delta u^i)(\omega d^i), w \rangle| \quad (46)$$

$$\begin{aligned} &\leq \int_{\Omega} |[\bar{\rho}(\Delta u^i + \omega d^i) D_p B(\Delta u^i + \omega d^i) - \bar{\rho}(\Delta u^i) D_p B(\Delta u^i) - \bar{\rho}(\Delta u^i) D_p B(\omega d^i)]^T B w| dx \\ &= \int_{\Omega} |\bar{\rho}(\Delta u^i + \omega d^i) - \bar{\rho}(\Delta u^i)| |[D_p B(\Delta u^i + \omega d^i)]^T B w| dx \\ &\leq \nu_0 K \|\omega d^i\|_E \|\Delta u^{i+1}\|_E \|w\|_E. \end{aligned} \quad (47)$$

Thus,

$$\|\bar{\mathcal{F}}(\Delta u^i + \omega d^i) - \bar{\mathcal{F}}(\Delta u^i) - A_{ep}(\Delta u^i)(\omega d^i)\|_{-E} \leq \nu_0 K \|\Delta u^{i+1}\|_E \omega \|A_e d^i\|_{-E} \quad (48)$$

Now, we estimate the reduction of the residual.

$$\begin{aligned} \|R(\Delta u^{i+1})\|_{-E} &= \|R(\Delta u^i + \omega d^i)\|_{-E} \\ &\leq \|(1 - \omega)R(\Delta u^i)\|_{-E} + \|\omega[R(\Delta u^i) + A_{ep}(\Delta u^i)d^i]\|_{-E} \\ &\quad + \|R(\Delta u^i + \omega d^i) - R(\Delta u^i) - A_{ep}(\Delta u^i)(\omega d^i)\|_{-E} \\ &\leq |1 - \omega| \|R(\Delta u^i)\|_{-E} + \omega \eta \|R(\Delta u^i)\|_{-E} + \omega \tilde{\zeta}_i \|A_e d^i\|_{-E}, \\ &\quad \tilde{\zeta}_i = \nu_0 K \|\Delta u^{i+1}\|_E. \end{aligned} \quad (49)$$

The inequality

$$|\langle (A_e - A_{ep}(\Delta u^i))d^i, w \rangle| = \left| \int_{\Omega} \rho(\Delta u^i) (D_p B d^i)^T B w dx \right| \leq \nu_0 \|d^i\|_E \|w\|_E,$$

implies that

$$\|(A_e - A_{ep}(\Delta u^i))d^i\|_{-E} \leq \nu_0 \|A_e d^i\|_{-E}$$

and therefore

$$\|A_e d^i\|_{-E} \leq \frac{1}{1 - \nu_0} \|A_{ep} d^i\|_{-E}. \quad (50)$$

Moreover, as a direct consequence of (43), we have

$$\|A_{ep} d^i\|_{-E} \leq (1 + \eta) \|R(\Delta u^i)\|_{-E}. \quad (51)$$

From (49), (50), (51), we obtain

$$\|R(\Delta u^{i+1})\|_{-E} \leq [|1 - \omega| + \omega(\tilde{\zeta}_i + \eta + \tilde{\zeta}_i \eta)] \|R(\Delta u^i)\|_{-E}, \quad (52)$$

$$\tilde{\zeta}_i = \frac{\nu_0 K}{1 - \nu_0} \|\Delta u^{i+1}\|_E. \quad (53)$$

Now, we can prove again that the ITS algorithm produces the sequence $\{\Delta u^i\}$ for which

$$\|\Delta u^i\|_E \leq \alpha, \quad (54)$$

$$\|R(\Delta u^{i+1})\|_{-E} \leq c \|R(\Delta u^i)\|_{-E} \quad (55)$$

with c from (44).

First, using (50), (51) and (45), we obtain

$$\begin{aligned}
\|\Delta u^1\|_E &\leq \|\Delta u^0\|_E + \omega \|d^0\|_E \\
&= \|\Delta u^0\|_E + \omega \|A_e d^0\|_{-E} \\
&\leq \|\Delta u^0\|_E + \omega \frac{1+\eta}{1-\nu_0} \|R(\Delta u^0)\|_{-E} \leq \alpha.
\end{aligned}$$

Thus, $\|\Delta u^1\|_E \leq \alpha$ and from (52), (53), we obtain the estimate (55).

Second, let (54), (55) be valid for all $i \leq j$. Then

$$\begin{aligned}
\|\Delta u^{j+1}\|_E &\leq \|\Delta u^j\|_E + \omega \|A_e d^j\|_{-E} \\
&\leq \|\Delta u^j\|_E + \omega \frac{1+\eta}{1-\nu_0} \|R(\Delta u^j)\|_{-E} \\
&\leq \|\Delta u^0\|_E + \omega \frac{1+\eta}{1-\nu_0} (1+c+\dots+c^j) \|R(\Delta u^0)\|_{-E} \\
&\leq \|\Delta u^0\|_E + \omega \frac{1+\eta}{1-\nu_0} \frac{1}{1-c} \|R(\Delta u^0)\|_{-E} \leq \alpha.
\end{aligned}$$

Hence, according to (52), (53) both (54), (55) are valid also for all $i \leq j+1$.

Now the proof of Theorem 3 can be completed in the same way as the proof of Theorem 2. \square

NOTE 5. The necessary conditions for convergence $\zeta < 1$ in Theorem 2 and $\bar{\zeta} < 1$ in Theorem 3 are equivalent.

For given α , we have $\bar{\zeta} < \zeta$ which shows faster convergence of the tangential stiffness method compared with the initial stiffness method. Moreover, for very small α , we obtain $\bar{\zeta} \ll \zeta$, i.e. the tangential stiffness method can be substantially faster for very small load increments.

8. Results for standard incremental techniques

Rigorous convergence proofs given in our paper require some regularization of the stress–strain relation for finite increments (8). Adopted δ -regularization, firstly introduced in [9], results in uniform Lipschitz continuity of the state multiplier $\bar{\rho}_\delta$ (see Lemma 1) and the nonlinear mapping $\bar{\mathcal{F}}$ (see Section 4). Practical use of δ -regularization would, of course, demand some further research concerning the suitable choice of the parameter δ and the accuracy of this kind of incremental finite element algorithm.

The standard computational plasticity algorithms, cf. [11], use another regularization technique. Its simplest form arises when the state multiplier ρ from (4) is replaced by $\rho_w = \rho_w(\sigma_h^k, \kappa_h^k, \Delta e_h)$,

$$\rho_w = \begin{cases} 0 & \text{if } P(1) \leq 0 \\ 1 - \theta_0 & \text{if } P(\theta_0) = 0 \end{cases} \quad \text{for some } \theta_0 \in (0, 1)$$

where $P(\theta) = P(\sigma_h^k + \theta D \Delta e_h, \kappa_h^k)$.

This simple weighting idea, introduced already in the earliest papers on the incremental finite element method cf. [10], can be further developed by taking D_p in $\sigma_h^k + \theta_0 D \Delta e_h$ instead of in σ_h^k or by some further enhancement of accuracy of the stress–strain relation by using the substepping technique [11].

The regularization by weighting or simply w -regularization results again in continuity of the stress–strain relation (at least for the von Mises plasticity with isotropic hardening) but not in uniform Lipschitz continuity. That is the reason why we have used the δ -regularization in our paper.

In our opinion, w -regularization is sufficient to guarantee the same convergence behaviour as described by our theory. As an example, we show here some results for a model problem, (see Fig. 1), and the inexact initial stiffness method.

In this model problem, the von Mises plasticity with elasticity modulus $E = 130$ MPa, Poisson ratio

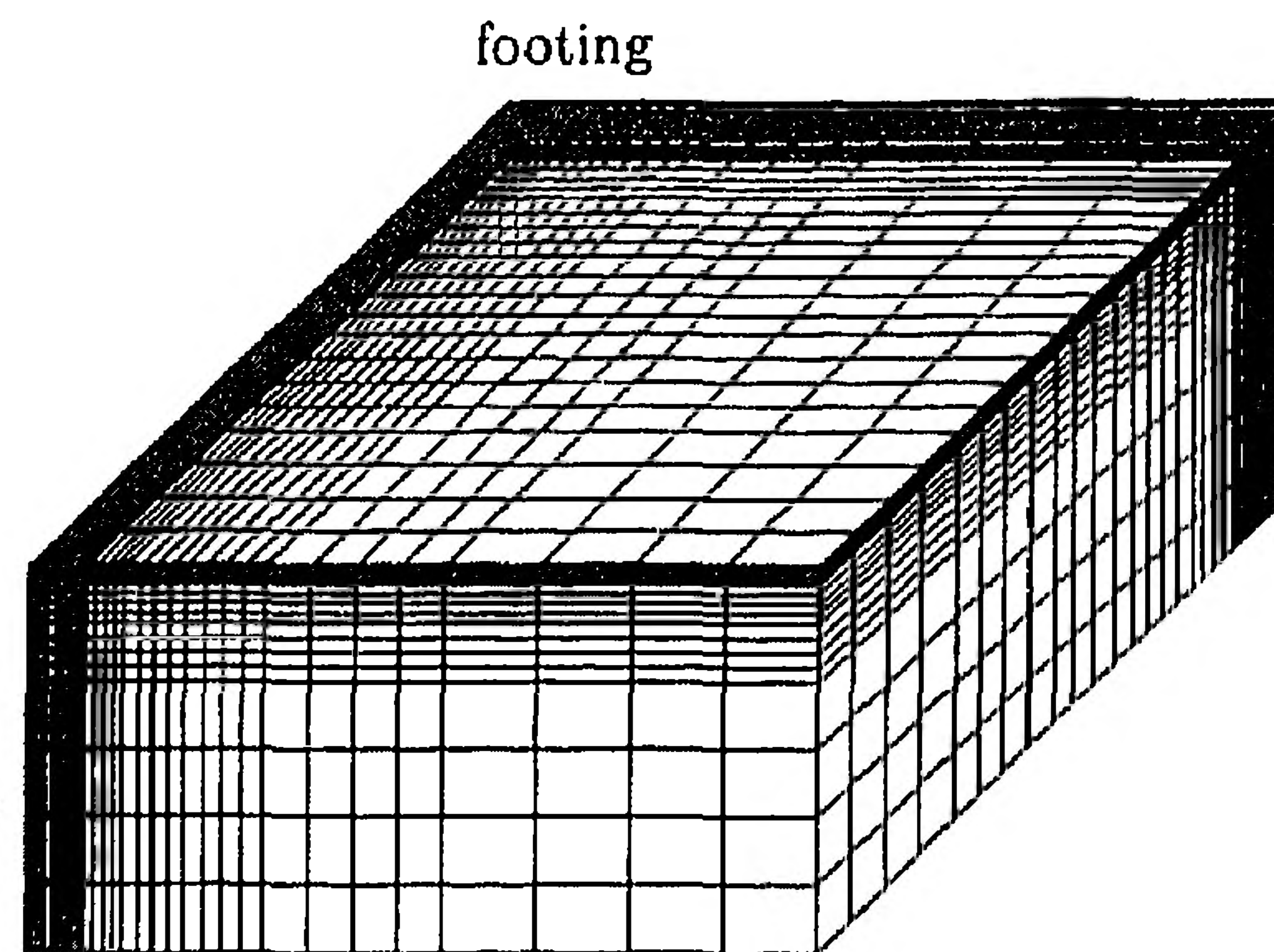


Fig. 1. Square footing problem.

Table 1
Total numbers of outer (inner) iterations

	$\eta = 0.01$	$\eta = 0.1$	ADAPT
NP = 5	45 (1379)	45 (667)	46 (399)
NP = 20	59 (1552)	60 (712)	63 (383)
NP = 100	62 (1319)	63 (506)	65 (474)

$\nu = 0.3$, isotropic hardening modulus $H' = 10.83$ MPa and the yield limit for uniaxial loading $\sigma_y = 0.58$ MPa is assumed. The problem is discretized in $33 \times 33 \times 17$ node grid which is locally refined under the footing. The domain is divided into bricks which are further divided each into six tetrahedra. The dimension of the arising nonlinear systems is 55 539. The first load increment covers the fully elastic behaviour, the size of the other local increments is at most $1/\text{NP}$ of the total load, where NP is a parameter.

The outer nonlinear iterations are stopped by the condition

$$\|R(\Delta u^i)\| \leq \varepsilon \|\Delta f^k\| \quad \text{with} \quad \varepsilon = 0.001.$$

Accuracy of the inner iterations is controlled in the Euclidian norm by the condition

$$\|A_e d^i + R(\Delta u^i)\| \leq \eta \|R(\Delta u^i)\|$$

with η given a priori or taken close to the reduction factor of the nonlinear iterations (the ADAPT technique). The inner iterations are performed by the conjugate gradient method with the displacement decomposition—incomplete factorization preconditioning (see [5]).

The results of computations can be seen from Table 1 which displays total numbers of inexact initial stiffness iterations (NIIS) and in brackets numbers of inner iterations (NPCG). These numbers determine the total computational work.

The displayed results correspond to using w -regularization in the form described in [11]. Without any regularization, the inexact initial stiffness method does not converge or converges extremely slowly for load increments given by $\text{NP} = 5$ and $\text{NP} = 20$. The convergence is again restored by taking sufficiently small load increments, e.g. for $\text{NP} = 100$ and $\eta = 0.1$ we obtain convergence with $\text{NIIS} = 75$ and $\text{NPCG} = 511$.

9. Conclusions

In Theorems 1, 2 and 3, we have proved the convergence of the inexact initial stiffness and the inexact tangential stiffness methods.

In Theorem 1 the convergence is guaranteed by both small load increments which give local monotonicity and Lipschitz continuity and the use of strong damping by the parameter ω .

The results of Theorems 2 and 3 are stronger. They concern also the inexact versions of the Newton-like methods and show that the choice of sufficiently small load increments is sufficient for the convergence. The additional damping by the parameter ω is unnecessary and therefore faster convergence rate can be expected. This is also in agreement with our numerical experiments and the experiments described in [7,3]. The convergence rate is especially high for problems with small plastic and transient zone, see Note 4.

Comparing the initial stiffness and the tangential stiffness method, we can see that the convergence range of both are the same and that the latter method converges faster, see Note 5.

Our convergence proofs give the convergence rate which depends on both the size of load increments and the size of the finite element mesh h . The convergence rate will not deteriorate with the refinement of the discretization if some kind of the stability condition is valid, e.g. if $\Delta t = O(h^{3/2})$ for linear finite elements.

Note finally that some further numerical examples and also comparisons with return mapping incremental algorithms can be found in [3] and [6].

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